# CONTRACTIBLE SUBGRAPHS AND MORITA EQUIVALENCE OF GRAPH $C^*$-ALGEBRAS

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ABSTRACT. In this paper we describe an operation on directed graphs which produces a graph with fewer vertices, such that the $C^*$-algebra of the new graph is Morita equivalent to that of the original graph. We unify and generalize several related constructions, notably delays and desingularizations of directed graphs.

## 1. INTRODUCTION

In recent years several authors have investigated certain constructions on directed graphs, derived from the theory of topological dynamics, which preserve the Morita equivalence class of the associated graph $C^*$-algebras [1],[2],[4],[6],[7]. Typically, these constructions have been viewed as enlargements of a graph which preserve its path structure. In [4], an attempt was made to unify and generalize some of the existing results on the subject, including the idea of a *delay*, which was examined in [6] and is the basis of the desingularization of [7]. It was noted [4, Remarks 4.6] that applying a delay replaces a vertex by a certain type of tree (called a *gantlet* in [6]), and that the Morita equivalence results for delays may still hold when a vertex is replaced by a more general tree.

In this paper we consider the reverse question: we aim to describe sufficient conditions for a subgraph of an arbitrary directed graph to be contractible, in the sense that its vertex set may be reduced to yield a graph whose $C^*$-algebra is Morita equivalent to that of the original graph. After a brief review of the established definitions and notations for graph algebras, we state and prove our main result, Theorem 3.1, which shows that any finite tree is contractible in the above sense, and that a similar construction may be applied to more general acyclic subgraphs. In particular, this theorem combines several of the separate results of [4] and may cover some further examples as well. Our proof follows closely the direct methods of [4], and makes use of a powerful theorem of [3], the gauge-invariant uniqueness theorem. Proposition 3.7 gives equivalent conditions to those of Theorem 3.1 which may make the theorem easier to apply, and which give some idea of what the contracted graph will look like. Section 4 discusses the relationship of our theorem to the existing results, and provides examples of graphs which are not contractible.

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## 2. Preliminaries

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets $E^0$ (vertices) and $E^1$ (edges), and maps $r, s : E^1 \to E^0$ describing the range and source of each edge. A vertex which emits no edges is called a *sink*; a vertex which emits infinitely-many edges is called an *infinite emitter*. A graph which contains no infinite emitters is called *row-finite*. Sinks and infinite emitters are collectively described as *singularities*, and we denote by $E^0_{sing}$ the set of all singularities in $E^0$. A *Cuntz-Krieger $E$-family* consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

(a) $S_e^* S_e = P_{r(e)}$,
(b) $S_e S_e^* \leq P_{s(e)}$, and
(c) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $v$ is not a singularity.

The graph $C^*$-algebra of $E$, denoted $C^*(E)$, is the universal $C^*$-algebra generated by a Cuntz-Krieger $E$-family $\{s_e, p_v\}$. For any graph $C^*$-algebra there is an action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ characterized by $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$ for $v \in E^0$ and $e \in E^1$. This *gauge action* is equivalent to the universal property of $C^*(E)$:

**Theorem 2.1.** [3, Theorem 2.1] *Let $E$ be a directed graph, $\{S_e, P_v\}$ be a Cuntz-Krieger $E$-family and $\pi : C^*(E) \to C^*(S_e, P_v)$ the homomorphism satisfying $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each $P_v$ is non-zero, and that there is a strongly continuous $\mathbb{T}$-action $\alpha$ on $C^*(S_e, P_v)$ such that $\alpha_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$. Then $\pi$ is faithful.*

Using the standard definitions and notations for paths in $E$ and the convention that a vertex is a path of length zero, we denote the set of finite paths by $E^*$, and the set of infinite paths by $E^\infty$. A finite path $\alpha$ of positive length is a *cycle* if $s(\alpha) = r(\alpha)$ and $s(\alpha_i) \neq s(\alpha_j)$ for $i \neq j$. An acyclic infinite path $\mu$ is a *tail* if each $s(\mu_i)$ emits only $\mu_i$ and each $r(\mu_i)$ receives only $\mu_i$. For $u, v \in E^0$ we say that $u \geq v$ if there is path in $E^*$ from $u$ to $v$. For $U \subseteq E^0$, we say $U \geq v$ if there exists $u \in U$ such that $u \geq v$. We define $v \geq U$ in a similar manner.

For $X \subset E^0$, we denote by $\Sigma H(X)$ the smallest saturated hereditary subset of $E^0$ containing $X$, as defined in [3, Remark 3.1]: $\Sigma H(X) := \bigcup_{n \geq 0} \Sigma_n(X)$, where $\Sigma_n(X)$ is defined inductively by

$$\Sigma_0(X) := X \cup \{w \in E^0 : X \geq w\},$$
$$\Sigma_{n+1}(X) := \Sigma_n(X) \cup \{w \in E^0 : 0 < |s^{-1}(w)| < \infty \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(X)\}.$$

[3, Section 3] demonstrated a correspondence between the saturated hereditary subsets of $E^0$ and the gauge-invariant ideals of $C^*(E)$, and [4, Lemma 2.2] gives a way to find full projections in $\mathcal{M}(C^*(E))$ by examining $\Sigma H(X)$ for suitable $X$.

## 3. Contractible subgraphs

**Theorem 3.1.** *Suppose $E$ is a directed graph with no tails, and suppose $G^0 \subset E^0$ such that $E^0_{sing} \subseteq G^0$ and the subgraph $T$ of $E$ defined by $T^0 := E^0 \setminus G^0$ and $T^1 := \{e \in E^1 : s(e), r(e) \in T^0\}$ is acyclic. Suppose that*

(a) *each vertex in $G^0$ is the source of at most one infinite path $\rho \in E^\infty$ such that $r(\rho_i) \in T^0$ for $i \geq 1$;*

*and that for each $\mu \in T^\infty$,*

(b) *$G^0 \geq s(\mu)$;*

(c) $|r^{-1}(s(\mu_i))| = 1$ *for all* $i$*; and*
(d) $e \in E^1, r(e) = s(\mu)$ *implies* $|s^{-1}(s(e))| < \infty$.

*Let* $G$ *be the graph with vertex set* $G^0$ *and one edge* $e_\beta$ *for each path* $\beta \in E^* \setminus E^0$ *with* $s(\beta), r(\beta) \in G^0$ *and* $r(\beta_i) \in T^0$ *for* $1 \leq i < |\beta|$*, such that* $s(e_\beta) = s(\beta)$ *and* $r(e_\beta) = r(\beta)$*. Then* $C^*(G)$ *is isomorphic to a full corner of* $C^*(E)$*.*

*Remarks* 3.2. *Conditions:* First note that any graph $C^*$-algebra can be approximated by the $C^*$-algebra of a graph with no tails, by replacing each tail with a sink as in [5, Lemma 1.2]. Now we specify an acyclic subgraph $T$ of $E$ containing none of the singularities of $E$, such that (a) holds, and every infinite path in $T$ satisfies conditions $(b)$–$(d)$. Let $v$ be a vertex on such a path $\mu$. Condition $(b)$ says that there is a path from $G^0$ to $v$. Condition $(c)$ says that $v$ receives exactly one edge. Condition $(d)$ says that if $u$ is a vertex which emits an edge with range $v$, then $u$ emits only finitely many edges. *Construction:* The idea is to replace each path $\beta$ through $T$ from $G^0$ to $G^0$ by a single edge $e_\beta$ having the same source and range as $\beta$. Since $T$ is acyclic and contains no singularities, it is reasonable to expect that this construction preserve the ideal structure of the graph algebra.

In order to prove Theorem 3.1 we shall first need to establish some important properties of the subgraph $T$:

**Lemma 3.3.** *Suppose* $v \in T^0$*. Then* $v \geq G^0$*.*

*Proof.* Suppose that no path in $E^*$ with source $v$ has range in $G^0$. Notice that we now have that $v \geq u \Longrightarrow u \not\geq G^0$. Now $v$ is not a sink, so it emits an edge $\mu_1$. By our assumption, $r(\mu_1) \in T^0$ and $r(\mu_1) \not\geq G^0$. Continuing in this manner gives a path $\mu = \mu_1\mu_2 \ldots \in T^\infty$. By Condition $(b)$ we can find a path $\alpha \in E^*$ such that $s(\alpha) \in G^0$, $r(\alpha) = v$ and $r(\alpha_i) \in T^0$ for $1 \leq i \leq |\alpha|$. Now the path $\mu$ is not a tail, and each $s(\mu_i)$ receives only one edge (by Condition $(c)$), so in particular there exists $k$ such that $s(\mu_k)$ emits an edge $\nu_1$ distinct from $\mu_k$. As before we must have $r(\nu_1) \in T^0$ and $r(\nu_1) \not\geq G^0$, so we can construct an infinite path $\nu = \nu_1\nu_2 \ldots \in T^\infty$ such that $s(\nu) = s(\mu_k)$. Now $\alpha\mu$ and $\alpha\mu_1 \ldots \mu_{k-1}\nu$ are distinct infinite paths which contradict Condition $(a)$. $\square$

**Lemma 3.4.** *For* $v \in E^0$ *define* $B_v = \{\beta \in E^* \setminus E^0 : s(\beta) = v, r(\beta) \in G^0$ *and* $r(\beta_i) \in T^0$ *for* $1 \leq i < |\beta|\}$*. Then:*

(a) *Suppose that* $v \in E^0$ *and that* $\alpha, \beta \in B_v$*. Then neither of* $\alpha$ *and* $\beta$ *is a proper extension of the other.*
(b) *Suppose* $\mu \in E^*$ *and* $s(\mu), r(\mu) \in G^0$*. Then* $\mu$ *is a product of paths in* $\bigcup_{v \in G^0} B_v$*.*
(c) $B_v = \emptyset \Longrightarrow v \in G^0$
(d) *Suppose* $v \in T^0$*. If* $|B_v|$ *is infinite then there exists* $\mu \in T^\infty$ *such that* $v = s(\mu)$*.*

*Proof.* (a) Suppose $\alpha = \beta\gamma$ for some $\gamma \in E^* \setminus E^0$. We have $|\beta| < |\alpha|$ and $r(\alpha_{|\beta|}) = r(\beta) \in G^0$, contradicting $\alpha \in B_v$.

(b) Let $\mu = \mu_1\mu_2 \ldots \mu_n$. The proof is by induction on $n$. The basis step $n = 1$ is given by definition of $B_v$. Now suppose that the assertion holds for paths of length less than $n$. If there exists $k$ such that $1 \leq k < n$ and $r(\mu_k) \in G^0$, then $\mu = (\mu_1 \ldots \mu_k)(\mu_{k+1} \ldots \mu_n)$ and hence $\mu$ is a product of paths in $\bigcup_{v \in G^0} B_v$ by the inductive hypothesis. If no such $k$ exists, then $\mu \in B_{s(\mu)}$ by definition, and $s(\mu) \in G^0$.

(c) Suppose that $v \in T^0$. By Lemma 3.3 there is a path $\alpha \in E^*$ with $s(\alpha) = v, r(\alpha) \in G^0$. Let $k$ be the first positive integer such that $r(\alpha_k) \in G^0$. Then $\alpha_1\alpha_2 \dots \alpha_k \in B_v$ by definition of $B_v$, hence $B_v$ is non-empty.

(d) Suppose $B_v$ is infinite. The vertex $v$ emits only finitely many edges, since $v \in T^0$. By the pigeonhole principle, at least one such edge $\mu_1$ must be the first edge in infinitely many $\beta$ in $B_v$. If $r(\mu_1) \in G^0$, then by definition we would have $\mu_1 \in B_v$, and part (a) then implies that $\mu_1$ can have no proper extension in $B_v$. Since we chose $\mu_1$ such that it had infinitely many extensions in $B_v$, we must have $r(\mu_1) \in T^0$. Notice that by definition, $\mu_1 \in T^1$. Since $B_{r(\mu_1)}$ is infinite, we can find an edge $\mu_2 \in s^{-1}(r(\mu_1)) \cap T^1$ such that $r(\mu_2) \in T^0$ and $B_{r(\mu_2)}$ is infinite. Repeating this construction gives a sequence $(\mu_1, \mu_2, \dots)$ of edges in $T^1$ such that for all $i$, $s(\mu_{i+1}) = r(\mu_i) \in T^0$. Then $\mu := \mu_1\mu_2 \dots$ is the infinite path required. □

**Lemma 3.5.** *Suppose that* $v \in E^0$ *and* $\sup\{|\beta| : \beta \in B_v\} = 1$*. Then* $B_v = s_E^{-1}(v)$*.*

*Proof.* Each path in $B_v$ must be a single edge, so we have $B_v \subseteq s_E^{-1}(v)$. For the reverse inclusion, suppose $e \in s_E^{-1}(v)$ and $r(e) \in T^0$. Lemma 3.4 (c) implies that $B_{r(e)}$ contains a path $\beta$, so $e\beta \in B_v$ is a path of length at least 2, a contradiction. Hence we must have $r(e) \in G^0$, and so $e \in B_v$ by definition. □

**Lemma 3.6.** *Suppose that* $v \in E^0$ *and* $B_v$ *is finite and non-empty. Then* $0 < |s_E^{-1}(v)| < \infty$*, and if* $\{s_e, p_w\}$ *is a Cuntz-Krieger E-family, we have*

$$p_v = \sum_{\beta \in B_v} s_\beta s_\beta^*$$

*Proof.* We begin by proving that $v$ is non-singular. Now $E^0_{sing} \subseteq G^0$, so we need only consider $v \in G^0$. Trivially $v$ cannot be a sink, so it remains to show that $|s_E^{-1}(v)| < \infty$. We do this by showing that each edge in $s_E^{-1}(v)$ has an extension in $B_v$. Suppose $s(e) = v$. If $r(e) \in G^0$ then by definition $e \in B_v$, so suppose $r(e) \in T^0$. There exists $\beta \in B_{r(e)}$ by Lemma 3.4 (c), so $e\beta \in B_v$ is an extension of $e$. Hence $|s_E^{-1}(v)| \leq |B_v| < \infty$.

Now define $N(v) := \sup\{|\beta| : \beta \in B_v\}$, and notice that this number is well-defined whenever $B_v$ is finite and non-empty. We shall prove the equality $p_v = \sum_{\beta \in B_v} s_\beta s_\beta^*$ by induction on $N(v)$.

Suppose $N(v) = 1$. Lemma 3.5 implies $B_v = s_E^{-1}(v)$, and the Cuntz-Krieger relation at $v$ gives $p_v = \sum_{f \in s_E^{-1}(v)} s_f s_f^* = \sum_{\beta \in B_v} s_\beta s_\beta^*$.

Now suppose that $N(v) = k$ and that $p_w = \sum_{\beta \in B_w} s_\beta s_\beta^*$ for any $w \in E^0$ with $1 \leq N(w) < k$. Then once again $v$ is non-singular, so

$$\begin{aligned} p_v = \sum_{f \in s^{-1}(v)} s_f s_f^* &= \sum_{f \in s^{-1}(v) \cap r^{-1}(G^0)} s_f s_f^* + \sum_{f \in s^{-1}(v) \cap r^{-1}(T^0)} s_f s_f^* \\ &= \sum_{\beta \in B_v \cap E^1} s_\beta s_\beta^* + \sum_{f \in s^{-1}(v) \cap r^{-1}(T^0)} s_f p_{r(f)} s_f^*. \end{aligned}$$

Consider an edge $f \in s^{-1}(v) \cap r^{-1}(T^0)$. We must have $B_{r(f)}$ non-empty by Lemma 3.4 (c). Each path $\alpha \in B_{r(f)}$ gives a path $f\alpha \in B_v$, so $B_{r(f)}$ must be finite and satisfy $N(r(f)) \leq k - 1$. Furthermore, every $\beta \in B_v$ with $|\beta| \geq 2$ has the form $g\alpha$ for some $g \in s^{-1}(v) \cap r^{-1}(T^0 \setminus G^0), \alpha \in B_{r(f)}$ : by definition

$r(\beta_1) \in s^{-1}(v) \cap r^{-1}(T^0)$ and $\beta_2 \dots \beta_{|\beta|} \in B_{r(\beta_1)}$. Thus for each such $f$ we can apply the inductive hypothesis to $r(f)$, giving

$$\begin{aligned} p_v &= \sum_{\beta \in B_v \cap E^1} s_\beta s_\beta^* + \sum_{f \in s^{-1}(v) \cap r^{-1}(T^0)} \left( \sum_{\alpha \in B_{r(f)}} s_{f\alpha} s_{f\alpha}^* \right) \\ &= \sum_{\beta \in B_v \cap E^1} s_\beta s_\beta^* + \sum_{\substack{\beta \in B_v, \\ |\beta| \geq 2}} s_\beta s_\beta^* \\ &= \sum_{\beta \in B_v} s_\beta s_\beta^*. \end{aligned}$$

This completes the proof by induction. □

*Proof of Theorem 3.1.* Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger $E$-family that generates $C^*(E)$. For $e_\beta \in G^1$ we define $T_{e_\beta} = s_\beta$, and for $v \in G^0$ we define $Q_v = p_v$.

The $Q_v$ are mutually orthogonal projections because the $p_v$ are. The $T_{e_\beta}$ are partial isometries because they are products of the partial isometries $s_f$ (recall the properties of Cuntz-Krieger families of partial isometries). To see that they have mutually orthogonal ranges, suppose $e_\alpha, e_\beta \in G^1$, $e_\alpha \neq e_\beta$. Then $\alpha, \beta \in \bigcup_{v \in G^0} B_v$ have the property that neither one is an extension of the other, by Lemma 3.4(a). [11, Lemma 1.1], which applies to infinite graphs, then implies $s_\alpha^* s_\beta = 0$, so $T_{e_\alpha} T_{e_\alpha}^* T_{e_\beta} T_{e_\beta}^* = s_\alpha s_\alpha^* s_\beta s_\beta^* = 0$, and $T_{e_\alpha}$ and $T_{e_\beta}$ have mutually orthogonal ranges. For $e_\beta \in G^1$ we have $T_{e_\beta}^* T_{e_\beta} = s_\beta^* s_\beta = p_{r(\beta)} = Q_{r(e_\beta)}$ and $T_{e_\beta} T_{e_\beta}^* = s_\beta s_\beta^* \leq p_{s(\beta)} = Q_{s(e_\beta)}$. Now suppose $v \in G^0$ is non-singular in $G$: that is, suppose $0 < |s_G^{-1}(v)| < \infty$. Since $s_G^{-1}(v)$ is equinumerous with $B_v$, $B_v$ is finite and non-empty. Lemma 3.6 then gives

$$Q_v = p_v = \sum_{\beta \in B_v} s_\beta s_\beta^* = \sum_{e_\beta \in s_G^{-1}(v)} T_{e_\beta} T_{e_\beta}^*.$$

Thus $\{T_{e_\beta}, Q_v\}$ is a Cuntz-Krieger $G$-family.

A slightly modified form of the argument of [5, Section 2] shows that there is a strongly continuous action $\alpha$ of $\mathbb{T}$ on $C^*(E)$ such that $\alpha_z(p_v) = p_v$ for all $v \in E^0$ and

$$\alpha_z(s_e) = \begin{cases} z s_e \text{ if } r(e) \in G^0 \\ s_e \text{ if } r(e) \in T^0 \end{cases}$$

Let $\{t_{e_\beta}, q_v\}$ be the canonical Cuntz-Krieger $G$-family. The universal property of $C^*(G)$ ensures the existence of a homomorphism $\pi$ of $C^*(G)$ onto $C^*(T_{e_\alpha}, Q_v)$ such that $\pi(t_{e_\beta}) = T_{e_\beta}$ and $\pi(q_v) = Q_v$ for all $e_\beta \in G^1$ and $v \in G^0$. If $\gamma$ is the canonical gauge action on $C^*(G)$, then $\pi \circ \gamma_z(q_v) = \alpha_z \circ \pi(q_v)$ for all $v \in G^0$ and $z \in \mathbb{T}$. Now fix $z \in \mathbb{T}$ and suppose $e_\beta \in G^1$. Then $\pi \circ \gamma_z(t_{e_\beta}) = \pi(z t_{e_\beta}) = z T_{e_\beta}$, and by definition of $B_v$, $\alpha_z \circ \pi(t_{e_\beta}) = \alpha_z(s_{\beta_1} \dots s_{\beta_{|\beta|}}) = s_{\beta_1} \dots s_{\beta_{|\beta|-1}} z s_{\beta_{|\beta|}} = z T_{e_\beta}$. Hence $\pi \circ \gamma = \alpha \circ \pi$ on all of $C^*(G)$. The gauge-invariant uniqueness theorem [3, Theorem 2.1] then implies that $\pi$ is an isomorphism of $C^*(G)$ onto $C^*(T_{e_\beta}, Q_v)$. We prove Theorem 3.1 by showing that $C^*(T_{e_\beta}, Q_v)$ is a full corner of $C^*(E)$.

By [5, Lemma 1.2(c)] the sum $\sum_{v \in G^0} p_v$ converges strictly to a projection $P \in \mathcal{M}(C^*(E))$. We claim that $C^*(T_{e_\beta}, Q_v) = PC^*(E)P$.

Note that for each $v \in G^0$ we have $Q_v \le P$, so $Q_v = PQ_vP \in PC^*(E)P$. We then have that for every $e_\beta \in G^1$,

$$T_{e_\beta} = Q_{s(e_\beta)} T_{e_\beta} Q_{r(e_\beta)} = PQ_{s(e_\beta)} T_{e_\beta} Q_{r(e_\beta)} P \in PC^*(E)P.$$

So $C^*(T_{e_\beta}, Q_v) \subseteq PC^*(E)P$ is easy.

Now fix $s_\mu s_\nu^* \in C^*(E)$. Then $Ps_\mu s_\nu^* P = \sum_{v,w\in G^0} p_v s_\mu s_\nu^* p_w$. Now $p_v s_\mu = 0$ unless $v = s(\mu)$, in which case $p_v s_\mu = s_\mu$. We can apply the same argument to $s_\nu^* p_w = (p_w s_\nu)^*$, so we have the following: Suppose that $Ps_\mu s_\nu^* P \neq 0$. Then $Ps_\mu s_\nu^* P = s_\mu s_\nu^*$, $s(\mu), s(\nu) \in G^0$, and $r(\mu) = r(\nu)$. Hence to show $Ps_\mu s_\nu^* P \in C^*(T_{e_\beta}, Q_v)$ it will suffice to consider the following three cases:

(1) $r(\mu) \in G^0$;
(2) $r(\mu) \in T^0$ and $B_{r(\mu)}$ is finite; and
(3) $r(\mu) \in T^0$ and $B_{r(\mu)}$ is infinite,

and to show in each case that $s_\mu s_\nu^* \in C^*(T_{e_\beta}, Q_v)$.

**Case 1.** By Lemma 3.4(b) we can write $\mu$ as a product $\alpha^1\alpha^2\dots\alpha^n$ of paths in $\bigcup_{v\in G^0} B_v$, so that $s_\mu = s_{\alpha^1}s_{\alpha^2}\dots s_{\alpha^n} \in C^*(T_{e_\beta}, Q_v)$. Similarly we can write $s_\nu = s_{\beta^1}s_{\beta^2}\dots s_{\beta^m} \in C^*(T_{e_\beta}, Q_v)$, so $s_\mu s_\nu^* \in C^*(T_{e_\beta}, Q_v)$.

**Case 2.** First notice that Lemma 3.4(c) implies that $B_{r(\mu)}$ is non-empty. Let $k := \max\{i : s(\mu_i) \in G^0\}$, and consider the paths $\rho := \mu_1 \dots \mu_{k-1}$ and $\gamma := \mu_k \dots \mu_{|\mu|}$. We can decompose $\mu$ as the product $\rho\gamma$ such that $s(\rho), r(\rho) \in G^0$, $s(\gamma) \in G^0$ and $r(\gamma_i) \in T^0$ for $1 \le i \le |\gamma|$. Similarly, we may write $\nu = \sigma\delta$ for some paths $\sigma, \delta \in E^*$ with the same properties as $\rho$ and $\gamma$, respectively. Case 1 shows that $s_\rho$ and $s_\sigma$ are in $C^*(T_{e_\beta}, Q_v)$, so to show $s_\mu s_\nu^* \in C^*(T_{e_\beta}, Q_v)$ it will be enough to show that $s_\gamma s_\delta^* \in C^*(T_{e_\beta}, Q_v)$. Since $B_{r(\mu)}$ is finite and non-empty, and $r(\mu) = r(\gamma) = r(\delta)$, we can use Lemma 3.6 to get

$$\begin{aligned} s_\gamma s_\delta^* &= s_\gamma p_{r(\mu)} s_\delta^* \\ &= s_\gamma \left( \sum_{\beta \in B_{r(\mu)}} s_\beta s_\beta^* \right) s_\delta^* \\ &= \sum_{\beta \in B_{r(\mu)}} s_{\gamma\beta} s_{\delta\beta}^* \\ &= \sum_{\beta \in B_{r(\mu)}} T_{e_{\gamma\beta}} T_{e_{\delta\beta}}^* \qquad \text{(since each } \gamma\beta, \delta\beta \in \bigcup_{v\in G^0} B_v). \end{aligned}$$

Thus $s_\mu s_\nu^*$ is a finite sum of elements of $C^*(T_{e_\beta}, Q_v)$, so $s_\mu s_\nu^* \in C^*(T_{e_\beta}, Q_v)$.

**Case 3.** (This is a combination of our method for Case 2 and the proof of [4, Theorem 4.2].) Lemma 3.4(d) implies that $r(\mu)$ is the source of some infinite path $\epsilon \in T^\infty$. By Condition ($b$) there exists a path $\alpha = \alpha_1 \dots \alpha_n \in E^*$ such that $s(\alpha) \in G^0$, $\alpha_2\alpha_3\dots\alpha_{|\alpha|} \in T^*$ and $r(\alpha) = r(\mu)$. For convenience we shall write $\alpha_0$ to denote the vertex $s(\alpha)$ viewed as a path of zero length, and for $0 \le i \le n$ we shall write $\gamma_i := \alpha_0 \dots \alpha_i$. Condition ($c$) implies that for each $i \ge 1$, $r(\alpha_i)$ receives only one edge, so we must have $\mu = \rho\alpha$ and $\nu = \sigma\alpha$ for some $\rho, \sigma \in E^*$. Once again Case 1 shows that $s_\rho, s_\sigma \in C^*(T_{e_\beta}, Q_v)$, so we need only prove $s_\alpha s_\alpha^* \in C^*(T_{e_\beta}, Q_v)$.

By our definition, $s_\alpha s_\alpha^* = s_{\gamma_n} s_{\gamma_n}^*$; our plan for proving $s_\alpha s_\alpha^* \in C^*(T_{e_\beta}, Q_v)$ is to reduce this product to $p_{s(\alpha)} - A$, where $A$ is a finite sum of elements of $C^*(T_{e_\beta}, Q_v)$. If we can do this, we will be done with Case 3: $s(\alpha) \in G^0$ implies

$p_{s(\alpha)} - A = Q_{s(\alpha)} - A \in C^*(T_{e_\beta}, Q_v)$. We perform this reduction recursively, as follows:

Suppose $0 \le k < n$. We shall show that $s_{\gamma_{k+1}} s^*_{\gamma_{k+1}} = s_{\gamma_k} s^*_{\gamma_k} - A_{k+1}$ where $A_{k+1}$ is a finite sum of elements of $C^*(T_{e_\beta}, Q_v)$. Since $\alpha_2 \dots \alpha_n \epsilon$ is a path in $T^\infty$, Condition $(d)$ implies that each $s^{-1}(s(\alpha_i))$ is finite. Suppose $s^{-1}(s(\alpha_{k+1})) = \{\alpha_{k+1}, f_1, \dots, f_m\}$. Then $s_{\alpha_{k+1}} s^*_{\alpha_{k+1}} = p_{s(\alpha_{k+1})} - \sum_{i=1}^m s_{f_i} s^*_{f_i}$, and so

$$s_{\gamma_{k+1}} s^*_{\gamma_{k+1}} = s_{\gamma_k} s_{\alpha_{k+1}} s^*_{\alpha_{k+1}} s^*_{\gamma_k} = s_{\gamma_k} p_{s(\alpha_{k+1})} s^*_{\gamma_k} - \sum_{i=1}^m (s_{\gamma_k} s_{f_i} s^*_{f_i} s^*_{\gamma_k}).$$

Since $s(\alpha_{k+1}) = r(\gamma_k)$, $s_{\gamma_k} p_{s(\alpha_{k+1})} s^*_{\gamma_k} = s_{\gamma_k} s^*_{\gamma_k}$ so we would like to show that each $s_{\gamma_k} s_{f_i} s^*_{f_i} s^*_{\gamma_k}$ is equal to a finite sum of elements of $C^*(T_{e_\beta}, Q_v)$. Fix $f \in s^{-1}(s(\alpha_{k+1})) \setminus \{\alpha_{k+1}\}$. If $r(f) \in G^0$, then the path $\gamma_k f$ is in $B_{s(\alpha)}$, and hence $s_{\gamma_k} s_{f_i} s^*_{f_i} s^*_{\gamma_k} = T_{e_{\gamma_k f}} T^*_{e_{\gamma_k f}} \in C^*(T_{e_\beta}, Q_v)$ as required. Now suppose $r(f) \notin G^0$. Then $B_{r(f)}$ is non-empty by Lemma 3.4(c); suppose it is infinite. Lemma 3.4(d) implies that $r(f) = s(\zeta)$ for some $\zeta \in T^\infty$. Then $\gamma_k f \zeta$ and $\alpha\epsilon$ are distinct paths which contradict Condition $(a)$. So $B_{r(f)}$ must in fact be finite. Hence we can apply Lemma 3.6 to give

$$\begin{aligned} s_{\gamma_k} s_f s^*_f s^*_{\gamma_k} &= s_{\gamma_k} s_f p_{r(f)} s^*_f s^*_{\gamma_k} \\ &= \sum_{\beta \in B_{r(f)}} s_{\gamma_k} s_f s_\beta s^*_\beta s^*_f s^*_{\gamma_k}. \end{aligned}$$

Each path $\gamma_k f \beta$ satisfies the requirements for membership of $B_{s(\alpha)}$, so we have

$$s_{\gamma_k} s_f s^*_f s^*_{\gamma_k} = \sum_{\beta \in B_{r(f)}} T_{e_{\gamma_k f \beta}} T^*_{e_{\gamma_k f \beta}}.$$

This is a finite sum of elements of $C^*(T_{e_\beta}, Q_v)$, which is precisely what we wanted.

We can now apply this reduction $n$ times to get $s_\alpha s^*_\alpha = s_{\gamma_0} s^*_{\gamma_0} - \sum_{k=1}^n A_k$, and since $s_{\gamma_0} = s_{\alpha_0} = p_{s(\alpha)}$ we are done with Case 3.

Now suppose $a = PAP \in PC^*(E)P$. We can find $A_n \in C^*(E)$ such that each $A_n$ is a finite linear combination of elements of the form $s_\mu s^*_\nu$ and $A_n \to A$. Cases 1, 2 and 3 show that each $PA_nP \in C^*(T_{e_\beta}, Q_v)$, and continuity of multiplication in $\mathcal{M}(C^*(E))$ implies $PA_nP \to PAP = a$. This gives $PC^*(E)P \subseteq C^*(T_{e_\beta}, Q_v)$, and it remains to show that $P$ is a full projection.

By [4, Lemma 2.2], to show that the projection $P$ is full we have only to show that $E^0 \subset \Sigma H(G^0)$. We have $G^0 \subset \Sigma H(G^0)$ by definition, so suppose $v \in T^0$, and note that Lemma 3.4(c) implies that $B_v$ is non-empty. If $B_v$ is infinite, then $v = s(\mu)$ for some $\mu \in T^\infty$ by Lemma 3.4(d); Condition $(b)$ then implies $G^0 \ge v$, so $v \in \Sigma H(G^0)$. Now suppose $B_v$ is finite, and note that Lemma 3.6 implies $0 < |s^{-1}(v)| < \infty$. Define $N(v) := \sup\{|\beta| : \beta \in B_v\}$ as in the proof of Lemma 3.6. We show that $v \in \Sigma H(G^0)$ by induction on $N(v)$.

Suppose $N(v) = 1$. Lemma 3.5 shows that every edge in $s^{-1}(v)$ has range in $G^0$, and since $0 < |s^{-1}(v)| < \infty$ we have $v \in \Sigma H(G^0)$. Now assume that for $w \in T^0$, $1 \le N(w) \le k$ implies that $w \in \Sigma H(G^0)$, and suppose $N(v) = k + 1 > 1$. For each edge $e \in s^{-1}(v)$, we must have either $r(e) \in G^0$ or $r(e) \in T^0$. Suppose $r(e) \in T^0$. Lemma 3.4(c) implies that $B_{r(e)} \ne \emptyset$, so $N(r(e)) \ge 1$. Now any path $\beta \in B_{r(e)}$ gives a path $e\beta \in B_v$ with length $|\beta| + 1$, so we must have $1 \le N(r(e)) \le k$. The

inductive hypothesis then implies $r(e) \in \Sigma H(G^0)$. Thus any edge $e$ with source $v$ has range in $\Sigma H(G^0)$, and since $v$ is non-singular, $v$ is then in $\Sigma H(G^0)$. It follows, by induction, that $T^0 \subset \Sigma H(G^0)$. Hence $E^0 \subset \Sigma H(G^0)$ as required, $P$ is a full projection in $\mathcal{M}(C^*(E))$, and $C^*(G)$ is isomorphic to a full corner of $C^*(E)$. $\square$

The conditions of Theorem 3.1 are based on a description of $T$, the subgraph to be contracted. It is possible to formulate equivalent conditions based on a description of $G^0$, the vertex set of the graph obtained by the contraction:

**Proposition 3.7.** *Suppose $E$ is a directed graph with no tails, and suppose $G^0 \subset E^0$ such that $E^0_{sing} \subseteq G^0$. Then $G^0$ satisfies the conditions of Theorem 3.1 if and only if it satisfies the following:*

1. *$\lambda \in E^*$ a cycle $\Longrightarrow s(\lambda_i) \in G^0$ for some $i$; and*
2. *Suppose $\mu, \nu \in E^\infty$ are distinct and acyclic; then*
   - *$(a')$ $s(\mu) = s(\nu) \in G^0 \Longrightarrow r(\mu_i) \in G^0$ or $r(\nu_i) \in G^0$ for some $i$.*
   - *$(b')$ Either $G^0 \geq s(\mu)$ or $s(\mu_i) \in G^0$ for some $i$;*
   - *$(c')$ $|r^{-1}(s(\mu))| > 1 \Longrightarrow s(\mu_i) \in G^0$ for some $i$;*
   - *$(d')$ $|s^{-1}(s(\mu))| = \infty \Longrightarrow r(\mu_i) \in G^0$ for some $i$.*

*Proof.* Suppose $\lambda \in E^*$ is a cycle. By definition of $T^1$, $\lambda \in T^* \iff s(\lambda_i) \in T^0$ for all $i$. Now any cycle in $T^*$ is a cycle in $E^*$, so the subgraph $T$ is acyclic if and only if (1) holds. Now each of Conditions $(a)$–$(d)$ in Theorem 3.1 is equivalent to the corresponding condition in the proposition. For example, suppose $(b)$ holds, and let $\mu \in E^\infty$ with $s(\mu_i) \in T^0$ for all $i$. Then by definition we have $\mu \in T^\infty$, so $(b)$ implies $G^0 \geq s(\mu)$, giving $(b')$. Conversely, suppose $(b')$ holds, and let $\mu \in T^\infty$. Then again by definition of $T^1$ we cannot have any $s(\mu_i) \in G^0$, so $G^0 \geq s(\mu)$, giving $(b)$. $\square$

## 4. Examples

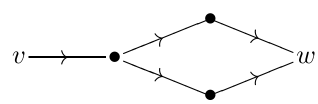
(i) For a graph $E$ with no tails, the desingularization $F$ of $E$, described in [7], is obtained by adding a tail at each infinite-emitter $v_0 \in E^0$ and distributing the edges in $s^{-1}(v_0)$ along this tail, such that the resulting graph is row-finite. It is straightforward to check that each such tail is acyclic, non-singular and satisfies Conditions $(a)$–$(d)$ of Theorem 3.1, and hence with $G^0 = E^0$, Theorem 3.1 gives Morita equivalence of $C^*(E)$ with $C^*(F)$ as in [7, Theorem 2.11].

(ii) [2, Section 5] examined a relation on directed graphs called *elementary strong shift equivalence*. Two graphs $E_1$ and $E_2$ are elementary strong shift equivalent via $E_3$ if $E_3$ is a bipartite graph whose vertex set $E^0_3$ is the disjoint union $E^0_1 \cup E^0_2$ such that the paths of length 2 in $E^*_3$ with source and range in $E^0_i$ are in one-to-one correspondence with the edges in $E_i$. It was shown ([2, Theorem 5.2]) that for row-finite graphs with no sinks, elementary strong shift equivalence implies Morita equivalence of the associated graph algebras. Setting $E = E_3$ and $G^0 = E^0_i$ for $i = 1, 2$ in Theorem 3.1 gives the same result. (To see that this is an applicable choice of $G^0$, notice that $E^0_{sing} = \emptyset$ and that the corresponding subgraph $T$ has no edges.)

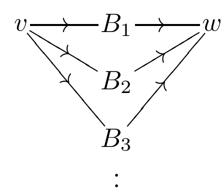
(iii) An out-delay $d_s(E)$ of a graph $E$ as described in [4] is obtained by adding some subpath of a tail (called a *gantlet* in [6]), possibly of length zero or $\infty$, to each vertex $v_0 \in E^0$ and distributing the edges in $s^{-1}(v_0)$ along this path. It can be seen that an out-delay is strictly proper, as defined in [4], if and only if the graph $T$ defined as the union of all the added gantlets satisfies Conditions $(a)$–$(d)$

of Theorem 3.1. Now taking $G^0 = E^0$, our Theorem 3.1 gives Theorem 4.2 of [4]. In a similar way Theorem 3.1, applied to an in-delay $d_r(E)$ of a graph $E$, gives [4, Theorem 4.5] when $G^0 = E^0$. The equivalence theorem for in-splittings (Corollary 5.4 of [4]) now follows from [4, Theorem 5.3] and Theorem 3.1.

(iv) Theorem 3.1 may be more general than the results of [4]; in particular, it covers situations where it may not be obvious that a finite sequence of delays and splittings give the required reduction. For example, denote by $B_n$ the binary tree with $n$ generations and all edges directed toward the leaves. Now let $v \longrightarrow B_n \longrightarrow w$ denote the graph with one edge from $v$ to the root of $B_n$, and one edge from each leaf of $B_n$ to $w$. For example, for $n = 2$ the graph is




Now consider the following graph $E$:

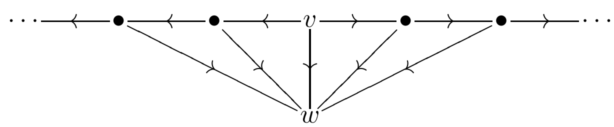



Letting $G^0 := \{v, w\}$, it can be seen that the conditions of Theorem 3.1 are satisfied and hence that $C^*(E)$ is Morita equivalent to $C^*(v \xrightarrow{\infty} w)$. It is not obvious that this result could be deduced from finitely-many applications of the results in [4]. Indeed, it seems reasonable to assume that the smallest number $r_n$ of applications of those results required to deduce the equivalence of $C^*\left(v \longrightarrow B_n \longrightarrow w\right)$ and $C^*(v \xrightarrow{2^{n-1}} w)$ should increase without bound as $n$ tends to infinity, and that the number of such applications required to deduce equivalence of $C^*(E)$ and $C^*(v \xrightarrow{\infty} w)$ should exceed every $r_n$.

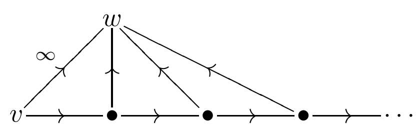
(v) In [9] the $C^*$-algebra $C(L_q(p; m_1, \ldots, m_n))$ of continuous functions on the quantum lens space $L_q(p; m_1, \ldots, m_n)$ was defined as the fixed point algebra $C(S_q^{2n-1})^{\tilde{\Lambda}}$ of a certain action of $\mathbb{Z}_p$ on the $C^*$-algebra of continuous functions on the odd-dimensional quantum sphere $S_q^{2n-1}$. It was shown in [8, Theorem 4.4] that $C(S_q^{2n-1})$ is isomorphic to the $C^*$-algebra of a directed graph $L_{2n-1}$. If $\Lambda$ is the $\mathbb{Z}_p$-action on this graph algebra corresponding to $\tilde{\Lambda}$, then [10, Corollary 2.5] shows that the crossed product $C^*(L_{2n-1}) \times_\Lambda \mathbb{Z}_p$ is itself the $C^*$-algebra of a certain graph, called the *skew product* graph. In [9, Theorem 2.5], the fixed point algebra corresponding to $\Lambda$ was also realized as a graph algebra $C^*\left(L_{2n-1}^{(p;m_1,\ldots,m_n)}\right)$. It can be seen that the graph $L_{2n-1}^{(p;m_1,\ldots,m_n)}$ may be obtained from the skew product graph $L_{2n-1} \times_c \mathbb{Z}_p$ by a contraction as in Theorem 3.1 (indeed, take $G^0 = (L_{2n-1})^0 \times \{0\}$). It is likely that this result can be generalized, so that the fixed point algebras corresponding to certain actions of finite groups on graph algebras may themselves be realized as

graph algebras, with the graph in question being obtained from the skew product graph by a contraction as in this example.

(vi) The following examples illustrate cases where Theorem 3.1 is not applicable. First, consider the following graph $E$:



If we relax Condition $(a)$, we can take $G^0 = \{v, w\}$ and deduce Morita equivalence of $C^*(E)$ and $C^*(v \xrightarrow{\infty} w)$. However, using the results of [3] it can be seen that $C^*(E)$ has 3 non-trivial ideals, while $C^*(v \xrightarrow{\infty} w)$ has only one. Now consider the following graph $F$:



Relaxing Condition $(d)$ and taking $G^0 = \{v, w\}$ gives Morita equivalence of $C^*(F)$ and $C^*(v \xrightarrow{\infty} w)$, which is again contradicted by counting the saturated hereditary subsets of $F^0$.

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